DigitalCommons @WayneState

Journal of Modern Applied Statistical Methods

Volume 5 | Issue 2 Article 31

11-1-2005

JMASM25: Computing Percentiles of Skew-Normal Distributions

Sikha Bagui
The University of West Florida, Pensacola, bagui@uwf.edu

Subhash Bagui
The University of West Florida, Pensacola, sbagui@uwf.edu

Journal of Modern Applied Statistical Methods
November, 2006, Vol. 5, No. 2, 575-588

1538 – 9472/06/$95.00

JMASM25: Computing Percentiles of Skew-Normal Distributions

Sikha Bagui Subhash Bagui
The University of West Florida

An algorithm and code is provided for computing percentiles of skew-normal distributions with parameter λ using Monte Carlo methods. A critical values table was created for various parameter values of λ at various probability levels of α. The table will be useful to practitioners as it is not available in the literature.

Key words: Skew normal distribution, critical values, visual basic.

Introduction

The skew normal (SN) class of densities is a family of densities that includes the normal density, with an extra parameter than the normal one to regulate skewness. In other words, SN class of densities extends the normal model by allowing an extra shape parameter to account for skewness. Let Z be a standard normal random variable with probability density function (pdf). ϕ and cumulative distribution function (cdf) Φ. A random variable Z' is said to have the one-parameter skew-normal distribution with parameter λ, (say, $Z' \sim SN(\lambda)$), if it has the following density function

$$f_{Z'}(z;\lambda) = 2\phi(z)\Phi(\lambda z), -\infty < z < \infty \quad (1)$$

where λ is a real valued parameter. When $\lambda = 0$, one gets the standard normal density, for $\lambda > 0$, one gets the positively skewed distribution, and for $\lambda < 0$, one gets the negatively skewed distribution. Graphs of skew-normal densities for different values of λ are shown in Figure 1.

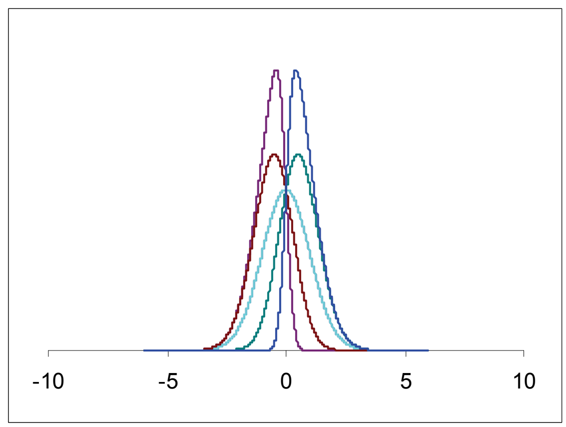

Figure 1: Skew-Normal Plot with $\lambda = -4, \ -1, \ 0, \ 1, \ 4$

The skew-normal distributions first indirectly appeared in Roberts (1966), O'Hagan and Leonard (1976), and Aigner and Lovell (1977). The landmark article by Azzalini (1985, 1986) gave a systematic treatment of this distribution and studied its fundamental properties. Recent work on characterization of skew-normal distribution can be found in Arnold and Lin (2004) and Gupta, Nguyen, and Sanqui (2004a). Azzalini and Dalla (1996), Azzalini and Capitanio (1999), and Gupta, Gonzalez-Farias, and Dominguez-Molina (2004b) investigated the multivariate extensions of this distribution. There is always a growing interest in modeling non-normal distributions. This

Sikha Bagui is an Assistant Professor in the Department of Computer Science. Her areas of research are database and database design, data mining, pattern recognition, and statistical computing. email: bagui@uwf.edu. Subhash Bagui is Professor of Statistics at the University of West Florida. His areas of research are statistical classification and pattern recognition, bio-statistics, construction of designs, tolerance regions, and statistical computing. email:sbagui@uwf.edu

model can be very useful to capture non-gaussian behavior of the data. It already proved to be useful in modeling real data sets (Azzalini & Captanio, 1999). To capture non-gaussian behavior of the data, Aigner and Lovell (1977) used this density in stochastic frontier models, Adcock and Shutes (1999) used it in portfolio selection of financial assets, and Bartolucci, De Luca, and Loperfido (2000) used it in detection of skewness in stock returns. Below some important properties of $SN(\lambda)$ are stated without proof. Let

$$\delta = \frac{\lambda}{\sqrt{1+\lambda^2}}$$

and

$$\Phi_\lambda(z) = \int_{-\infty}^{z} f_{Z'}(t;\lambda)dt .$$

1. $SN(0)$ has the density $N(0,1)$.
2. If $Z' \sim SN(\lambda)$, then $-Z' \sim SN(-\lambda)$.
3. $f_{Z'}(z;\lambda)$ is strongly unimodal, i.e. $\log f_{Z'}(z;\lambda)$ is a concave function of z.
4. $\Phi_\lambda(-z) = 1 - \Phi_{-\lambda}(z)$, i.e. $f_{Z'}(z;\lambda)$ is a mirror image of $f_{Z'}(z;-\lambda)$.
5. If $Z' \sim SN(\lambda)$, then $Z'^2 \sim \chi_1^2$ (chi-square with one d.f.).
6. If $Z' \sim SN(\lambda)$, then Z' has the m.g.f. $M_{Z'}(t) = 2\exp(\frac{t^2}{2})\Phi(\delta t)$.
7. If $Z' \sim SN(\lambda)$, then $E(Z') = \sqrt{\frac{2}{\pi}}\delta$ and $\mathrm{Var}(Z') = 1 - \frac{2}{\pi}\delta^2$.
8. Let Z_1 and Z_2 be independent $N(0,1)$ random variables, then $Z' = \delta|Z_1| + \sqrt{1-\delta^2}Z_2$ $\sim SN(\lambda)$.
9. Let Z_1 and Z_2 be independent $N(0,1)$ random variables. Set
$$Z' = \begin{cases} Z_2 & \text{if } Z_1 < \lambda Z_2 \\ -Z_2 & \text{if } Z_1 > \lambda Z_2. \end{cases}$$

Then $Z' \sim SN(\lambda)$.

10. Let (Z_1, Z_2) be a bivariate normal with $E(Z_1) = E(Z_2) = 0$, $\mathrm{Var}(Z_1) = \mathrm{Var}(Z_2) = 1$, and $\mathrm{corr}(Z_1, Z_2) = \rho$. Set $Z' = \max(Z_1, Z_2)$. Then $Z' \sim SN(\lambda)$, where $\lambda = \sqrt{\frac{1-\rho}{1+\rho}}$.

Property (4) tells us that $f_{Z'}(z;\lambda)$ is a mirror image of $f_{Z'}(z;-\lambda)$. So it is sufficient to find percentiles only for positive values of λ. For negative values of λ, one may use property (4) to derive the percentiles. In order to simulate skew-normal random variate one may use either properties (8), or (9), or (10). It is also important that sufficient number of random variates must be used to model the distribution of Z' adequately. Below we extend one parameter skew-normal distribution to three parameters skew-normal distributions.

Definition. (*Three-parameter skew-normal*). A random variable X' is said to have the three-parameter skew-normal distribution (denoted by $X' \sim SN(\mu, \sigma, \lambda)$) if $X' = \mu + \sigma Z'$, where $Z' \sim SN(\lambda)$, i.e. if X' has the following density function

$$f_{X'}(x;\lambda) = \frac{2}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)\Phi\left(\lambda\frac{x-\mu}{\sigma}\right), -\infty < x < \infty \tag{2}$$

Methodology

Property (8) will be used in order to generate skew-normal random variate Z'. Before this property is used, standard normal variates must be simulated. The following theorem explains how to generate standard normal variates.

Theorem.

(Box & Muller, 1958). Let U_1 and U_2 be independent random variates from uniform $(0,1)$ (i..e. U_1, $U_2 \sim U(0,1)$), then the variates $Z_1 = (-2\ln U_1)^{\frac{1}{2}} \cos 2\pi U_2$; $Z_2 = (-2\ln U_1)^{\frac{1}{2}} \sin 2\pi U_2$ are independent standard normal variates.

Proof.

The joint density of U_1 and U_2 is given by $f_{U_1,U_2}(u_1,u_1) = 1$. Also, from the above transformation note that $u_1 = e^{-\frac{1}{2}(z_1^2+z_2^2)}$ and $u_2 = \frac{1}{2\pi}\tan^{-1}\frac{z_2}{z_1}$. Thus, the Jacobian of this transformation is given by $J = -\frac{1}{2\pi}e^{-\frac{1}{2}(z_1^2+z_2^2)}$. Hence, the joint density of Z_1 and Z_2 is given by

$$
\begin{aligned}
f_{Z_1,Z_2}(z_1,z_2) &= f_{U_1,U_2}(u_1,u_1) \times |J| \\
&= \frac{1}{2\pi}e^{-\frac{1}{2}(z_1^2+z_2^2)} .
\end{aligned}
\tag{3}
$$

The equation (3) confirms that Z_1 and Z_2 is independent standard normal variates.

Now using the above theorem and the property (8), the skew-normal variates Z' shall be generated. The algorithm is as follows.

Algorithm

1. Define $\delta = \frac{\lambda}{\sqrt{1+\lambda^2}}$
2. Generate two independent random variates U_1 and U_2 from $U(0,1)$.
3. Set $Z_1 \leftarrow (-2\ln U_1)^{\frac{1}{2}} \cos 2\pi U_2$ and $Z_2 \leftarrow (-2\ln U_1)^{\frac{1}{2}} \sin 2\pi U_2$.
4. Deliver $Z' = \delta|Z_1| + \sqrt{1-\delta^2}Z_2$. □

This process is repeated until the density of Z' is adequately modeled. The code is written in Visual Basic. After a sufficient number of runs the program will return percentiles corresponding to the probability level $\alpha =$ 0.01, 0.02, 0.025, 0.05, 0.1, 0.99, 0.095, 0.975, 0.98, 0.99 for a given λ value. For $\lambda = 0.01$ to 50, in Table 1A, the percentiles of Z' are provided for $\alpha = 0.01$, 0.02, 0.025, 0.05, 0.1; and in Table 1B, the percentiles are provided for $\alpha = 0.99$, 0.095, 0.975, 0.98, 0.99. Let the notation $Z'_\alpha(\lambda)$ denote the $\alpha(100)\%$ percentile of Z' for a given value of λ.

Percentiles for negative λ:

Let $\lambda > 0$, then $Z'_\alpha(-\lambda)$ can be derived by means of $Z'_\alpha(-\lambda) = -Z'_{1-\alpha}(\lambda)$, as $f_{Z'}(z;\lambda)$ is a mirror image of $f_{Z'}(z;-\lambda)$.

Percentiles of X':

For the three-parameter skew normal r.v. $X' \sim SN(\mu,\ \sigma,\ \lambda)$, let $X'_\alpha(\lambda)$ denote the $\alpha(100)\%$ percentile of X' for a given value of λ. Since $X' = \mu + \sigma Z'$ is a monotonically increasing transformation, the percentile $X'_\alpha(\lambda)$ of X' can be obtained from the equation

$$
X'_\alpha(\lambda) = \mu + \sigma Z'_\alpha(\lambda) . \tag{4}
$$

Table 1A. Percentiles of Skew-normal distributions

$\lambda\downarrow$ $\vert\alpha\rightarrow$	0.01	0.02	0.025	0.04	0.05	0.10
0.01	-2.316601	-2.043600	-1.949637	-1.740196	-1.634294	-1.270585
0.02	-2.308165	-2.035310	-1.941678	-1.732143	-1.626163	-1.262473
0.03	-2.299909	-2.027109	-1.933209	-1.723888	-1.617982	-1.254325
0.04	-2.291742	-2.018600	-1.924809	-1.715525	-1.609591	-1.246031
0.05	-2.282902	-2.010202	-1.916329	-1.707010	-1.601178	-1.237716
0.06	-2.273634	-2.001495	-1.907718	-1.698454	-1.592648	-1.229322
0.07	-2.264764	-1.992613	-1.899061	-1.689889	-1.584055	-1.220915
0.08	-2.255799	-1.983832	-1.890096	-1.681037	-1.575384	-1.212358
0.09	-2.246697	-1.974694	-1.881151	-1.672253	-1.566573	-1.203799
0.10	-2.237531	-1.965540	-1.872131	-1.663274	-1.557718	-1.195168
0.15	-2.189265	-1.918519	-1.825301	-1.617421	-1.512277	-1.151156
0.20	-2.137967	-1.869018	-1.776368	-1.569685	-1.465172	-1.106155
0.25	-2.084749	-1.817640	-1.725684	-1.520615	-1.416825	-1.060535
0.30	-2.029456	-1.764838	-1.673795	-1.470527	-1.367758	-1.014637
0.35	-1.972884	-1.711116	-1.620986	-1.420003	-1.318257	-0.968773
0.40	-1.915085	-1.656844	-1.567945	-1.369303	-1.268771	-0.923367
$\lambda\downarrow$ $\vert\alpha\rightarrow$	0.01	0.02	0.025	0.04	0.05	0.10
0.45	-1.857159	-1.602308	-1.514616	-1.318676	-1.219467	-0.878647
0.50	-1.79914	-1.548298	-1.461832	-1.268712	-1.170971	-0.834859
0.55	-1.741834	-1.494818	-1.409644	-1.219545	-1.123254	-0.792200
0.60	-1.684862	-1.442066	-1.358373	-1.171422	-1.076717	-0.750845
0.65	-1.629129	-1.390497	-1.308256	-1.124578	-1.031460	-0.710951
0.70	-1.574326	-1.340210	-1.259576	-1.079053	-0.987594	-0.672611
0.75	-1.521160	-1.291533	-1.212292	-1.035095	-0.945313	-0.635839
0.80	-1.469659	-1.244274	-1.166618	-0.992756	-0.904630	-0.600690
0.85	-1.419618	-1.198740	-1.122574	-0.951997	-0.865531	-0.567114
0.90	-1.371167	-1.154904	-1.080178	-0.912938	-0.828105	-0.535187
0.95	-1.324828	-1.112776	-1.039482	-0.875538	-0.792291	-0.504804
1.0	-1.280150	-1.072379	-1.000580	-0.839787	-0.758158	-0.475989
1.1	-1.195920	-0.996480	-0.927569	-0.773014	-0.694478	-0.422673
1.2	-1.118556	-0.927121	-0.860912	-0.712349	-0.636756	-0.374875
1.3	-1.047759	-0.863910	-0.800243	-0.657286	-0.584519	-0.331973
1.4	-0.982884	-0.806225	-0.744969	-0.607346	-0.537221	-0.293562
1.5	-0.923732	-0.753687	-0.694679	-0.562104	-0.494470	-0.259078
1.6	-0.869510	-0.705780	-0.648922	-0.521012	-0.455715	-0.228142
1.7	-0.819759	-0.661996	-0.607135	-0.483662	-0.420568	-0.200318
1.8	-0.774230	-0.621985	-0.569028	-0.449680	-0.388656	-0.175231
1.9	-0.732466	-0.585328	-0.534144	-0.418682	-0.359602	-0.152622
2.0	-0.694041	-0.551756	-0.502219	-0.390393	-0.333112	-0.132153
2.1	-0.658558	-0.520897	-0.472897	-0.364488	-0.308919	-0.113589
2.2	-0.625889	-0.492486	-0.445939	-0.340752	-0.286783	-0.096753
2.3	-0.595671	-0.466286	-0.421117	-0.318922	-0.266441	-0.081409
2.4	-0.567697	-0.442084	-0.398212	-0.298842	-0.247777	-0.067438
2.5	-0.541678	-0.419685	-0.376990	-0.280305	-0.203571	-0.054627
2.6	-0.517496	-0.398873	-0.357355	-0.263187	-0.214697	-0.042918
2.7	-0.495071	-0.379556	-0.339106	-0.247320	-0.200020	-0.032172
2.8	-0.474034	-0.361518	-0.322117	-0.232604	-0.186429	-0.022296

Table 1A. Continued

2.9	-0.454425	-0.344773	-0.306319	-0.218931	-0.173802	-0.013191
3.0	-0.436086	-0.329099	-0.291554	-0.206190	-0.162091	-0.004783
3.1	-0.418815	-0.314420	-0.277760	-0.194313	-0.151160	0.002981
3.2	-0.402685	-0.300694	-0.264865	-0.183231	-0.140979	0.010144
3.3	-0.387500	-0.287812	-0.252750	-0.172862	-0.131481	0.016818
3.4	-0.373191	-0.275705	-0.241405	-0.163161	-0.122593	0.022994
3.5	-0.359675	-0.264295	-0.230702	-0.154042	-0.114251	0.028741
3.6	-0.346930	-0.253540	-0.220636	-0.145471	-0.106434	0.034091
3.7	-0.334911	-0.243416	-0.211150	-0.137420	-0.099090	0.039075
3.8	-0.323507	-0.233839	-0.202214	-0.129843	-0.092200	0.043727
3.9	-0.312720	-0.224801	-0.193740	-0.122698	-0.085694	0.048080
4.0	-0.302460	-0.216232	-0.185747	-0.115946	-0.079572	0.052150
4.1	-0.292758	-0.208106	-0.178181	-0.109578	-0.073800	0.055953
4.2	-0.283507	-0.200385	-0.170970	-0.103532	-0.068333	0.059509
4.3	-0.274730	-0.193095	-0.164176	-0.097830	-0.063178	0.062874
4.4	-0.266349	-0.186124	-0.157692	-0.092408	-0.058288	0.066000
4.5	-0.258354	-0.179500	-0.151528	-0.087270	-0.053652	0.068956
4.6	-0.250720	-0.173180	-0.145664	-0.082397	-0.049265	0.071733
4.7	-0.243464	-0.167145	-0.140082	-0.077752	-0.045097	0.074366
4.8	-0.236517	-0.161429	-0.134749	-0.073343	-0.041141	0.076857
4.9	-0.229873	-0.155942	-0.129676	-0.069135	-0.037373	0.079159
5.0	-0.223494	-0.150707	-0.124817	-0.065139	-0.033789	0.081360
$\lambda\downarrow \quad \vert\alpha\rightarrow$	0.01	0.02	0.025	0.04	0.05	0.10
5.1	-0.217406	-0.145728	-0.120181	-0.061327	-0.030392	0.083439
5.2	-0.211567	-0.140914	-0.115759	-0.057684	-0.027137	0.085408
5.3	-0.205935	-0.136321	-0.111519	-0.054198	-0.024034	0.087269
5.4	-0.200537	-0.131905	-0.107427	-0.050879	-0.021072	0.089026
5.5	-0.195384	-0.127710	-0.103541	-0.047696	-0.018256	0.090667
5.6	-0.190408	-0.123636	-0.099800	-0.044650	-0.015548	0.092261
5.7	-0.185623	-0.119771	-0.096229	-0.041742	-0.012973	0.093766
5.8	-0.181004	-0.116017	-0.092782	-0.038952	-0.010507	0.095178
5.9	-0.176603	-0.112419	-0.089485	-0.036272	-0.008136	0.096523
6.0	-0.172317	-0.108956	-0.086301	-0.033701	-0.005888	0.097816
6.1	-0.168194	-0.105632	-0.083230	-0.031240	-0.003710	0.099030
6.2	-0.164207	-0.102420	-0.080282	-0.028865	-0.001611	0.100179
6.3	-0.160373	-0.099323	-0.077451	-0.026586	0.000375	0.101290
6.4	-0.156667	-0.096356	-0.074712	-0.024401	0.002299	0.102334
6.5	-0.153083	-0.093483	-0.072070	-0.022295	0.004149	0.103340
6.6	-0.149635	-0.090704	-0.069549	-0.020284	0.005903	0.104284
6.7	-0.146298	-0.088027	-0.067097	-0.018334	0.007612	0.105187
6.8	-0.143068	-0.085447	-0.064730	-0.016452	0.009265	0.106051
6.9	-0.139926	-0.082934	-0.062453	-0.014642	0.010826	0.106886
7.0	-0.136863	-0.080531	-0.060243	-0.012900	0.012341	0.107663
7.1	-0.133958	-0.078188	-0.058108	-0.011215	0.013789	0.108414
7.2	-0.131101	-0.075920	-0.056037	-0.009603	0.015194	0.109131
7.3	-0.128363	-0.073745	-0.054056	-0.008027	0.016553	0.109804

Table 1A. Continued

7.4	-0.125676	-0.071620	-0.052122	-0.006516	0.017858	0.110468
7.5	-0.123042	-0.069554	-0.050240	-0.005032	0.019123	0.111083
7.6	-0.120535	-0.067574	-0.048431	-0.003635	0.020326	0.111686
7.7	-0.118106	-0.065650	-0.046692	-0.002276	0.021503	0.112249
7.8	-0.115726	-0.063777	-0.044988	-0.000954	0.022632	0.112797
7.9	-0.113417	-0.061967	-0.043348	0.000319	0.023724	0.113323
8.0	-0.111195	-0.060210	-0.041749	0.001560	0.024790	0.113809
8.1	-0.109002	-0.058498	-0.040200	0.002743	0.025790	0.114317
8.2	-0.106879	-0.056837	-0.038692	0.003900	0.026798	0.114763
8.3	-0.104818	-0.055219	-0.037241	0.005040	0.027749	0.115201
8.4	-0.102825	-0.053661	-0.035826	0.006123	0.028656	0.115608
8.5	-0.100859	-0.052133	-0.034446	0.007154	0.029559	0.116018
8.6	-0.098962	-0.050651	-0.033115	0.008175	0.030416	0.116405
8.7	-0.097105	-0.049204	-0.031813	0.009179	0.031252	0.116774
8.8	-0.095290	-0.047811	-0.030556	0.010149	0.032062	0.117126
8.9	-0.093523	-0.046447	-0.029317	0.011070	0.032860	0.117473
9.0	-0.091821	-0.045127	-0.028133	0.011977	0.033618	0.117783
9.1	-0.090159	-0.043830	-0.026962	0.012845	0.034346	0.118100
9.2	-0.088516	-0.042576	-0.025842	0.013698	0.035065	0.118397
9.3	-0.086938	-0.041349	-0.024734	0.014532	0.035755	0.118687
9.4	-0.085379	-0.040158	-0.023666	0.015333	0.036421	0.118960
9.5	-0.083854	-0.038984	-0.022616	0.016118	0.037080	0.119222
9.6	-0.082381	-0.037854	-0.021604	0.016868	0.037714	0.119475
9.7	-0.080931	-0.036748	-0.020609	0.017608	0.038328	0.119719
9.8	-0.079553	-0.035683	-0.019654	0.018317	0.038917	0.119954
9.9	-0.078172	-0.034629	-0.018706	0.019018	0.039480	0.120186
10.0	-0.076817	-0.033597	-0.017784	0.019709	0.040050	0.120411
10.5	-0.070540	-0.028825	-0.013538	0.022821	0.042624	0.121345
11.0	-0.064912	-0.024585	-0.009749	0.025563	0.044873	0.122133
$\lambda \downarrow \quad \mid \alpha \rightarrow$	0.01	0.02	0.025	0.04	0.05	0.10
11.5	-0.059842	-0.020784	-0.006407	0.027966	0.046819	0.122770
12.0	-0.055255	-0.017392	-0.003412	0.030098	0.048531	0.123291
12.5	-0.051114	-0.014327	-0.000708	0.031980	0.050026	0.123727
13.0	-0.047339	-0.011552	0.001725	0.033664	0.051355	0.124079
13.5	-0.043889	-0.009055	0.003918	0.035157	0.052524	0.124365
14.0	-0.040724	-0.006747	0.005909	0.036500	0.053572	0.124596
14.5	-0.037822	-0.004656	0.007694	0.037699	0.054480	0.124805
15.0	-0.035151	-0.002762	0.009352	0.038784	0.055308	0.124958
15.5	-0.032685	-0.001014	0.010850	0.039764	0.056045	0.125093
16.0	-0.030383	0.000586	0.012218	0.040657	0.056694	0.125200
16.5	-0.028268	0.002073	0.013480	0.041444	0.057285	0.125281
17.0	-0.026302	0.003431	0.014648	0.042177	0.057812	0.125357
17.5	-0.024471	0.004693	0.015709	0.042825	0.058293	0.125411
18.0	-0.022750	0.005851	0.005851	0.016690	0.058715	0.125456

Table 1A. Continued

18.5	-0.021154	0.006931	0.017599	0.043967	0.059083	0.125477
19.0	-0.019658	0.007942	0.018437	0.044467	0.059422	0.125499
19.5	-0.018247	0.008911	0.019235	0.044918	0.059745	0.125521
20.0	-0.016935	0.009766	0.019949	0.045335	0.060028	0.125530
20.5	-0.015693	0.010567	0.020614	0.045729	0.060287	0.125547
21.0	-0.014518	0.011346	0.021244	0.046078	0.060514	0.125554
21.5	-0.013422	0.012052	0.021830	0.046386	0.060714	0.125561
22.0	-0.012383	0.012723	0.022374	0.046681	0.060905	0.125560
22.5	-0.011397	0.013344	0.022886	0.046948	0.061078	0.125566
23.0	-0.010471	0.013928	0.023348	0.047200	0.061233	0.125567
23.5	-0.009614	0.014482	0.023801	0.047428	0.061370	0.125573
24.0	-0.008769	0.014998	0.024219	0.047639	0.061500	0.125572
24.5	-0.007970	0.015477	0.024611	0.047831	0.061614	0.125574
25.0	-0.007208	0.015950	0.024969	0.048007	0.061716	0.125564
25.5	-0.006509	0.016391	0.025315	0.048170	0.061808	0.125573
26.0	-0.005832	0.016791	0.025634	0.048323	0.061892	0.125564
26.5	-0.005174	0.017176	0.025939	0.048463	0.061967	0.125552
27.0	-0.004559	0.017544	0.026223	0.048588	0.062036	0.125554
27.5	-0.003968	0.017898	0.026496	0.048705	0.062091	0.125542
28.0	-0.003411	0.018220	0.026747	0.048813	0.062153	0.125537
28.5	-0.002880	0.018530	0.026980	0.048919	0.062205	0.125533
29.0	-0.002364	0.018821	0.027205	0.049011	0.062245	0.125530
29.5	-0.001887	0.019102	0.027418	0.049094	0.062286	0.125524
30.0	-0.001412	0.019371	0.027628	0.049182	0.062326	0.125530
31.0	-0.000534	0.019867	0.028003	0.049323	0.062388	0.125514
32.0	-0.000270	0.020300	0.028323	0.049445	0.062445	0.125508
33.0	0.001017	0.020702	0.028611	0.049549	0.062486	0.125501
34.0	0.001689	0.021065	0.028878	0.049633	0.062532	0.125496
35.0	0.002328	0.021390	0.029116	0.049712	0.062557	0.125504
36.0	0.002914	0.021690	0.029332	0.049774	0.062578	0.125490
37.0	0.003455	0.021970	0.029521	0.049829	0.062591	0.125496
38.0	0.003950	0.022231	0.029697	0.049876	0.062610	0.125493
39.0	0.004439	0.022450	0.029851	0.049918	0.062614	0.125490
40.0	0.004874	0.022661	0.029990	0.049952	0.062626	0.125487
45.0	0.006671	0.023457	0.030504	0.050056	0.062642	0.125448
50.0	0.007977	0.023984	0.030829	0.050084	0.062631	0.125432

Table 1B. Percentiles of Skew-normal distributions

$\lambda\downarrow \quad \vert\alpha\rightarrow$	0.90	0.95	0.96	0.975	0.98	0.99
0.01	1.293420	1.656375	1.762111	1.971312	2.064999	2.337648
0.02	1.301256	1.664239	1.769951	1.979043	2.072866	2.344827
0.03	1.309050	1.671955	1.777612	1.986665	2.080348	2.352550
0.04	1.316720	1.679481	1.785125	1.994172	2.087801	2.360183
0.05	1.324280	1.686977	1.792677	2.001513	2.095286	2.367557
0.06	1.331719	1.694337	1.799951	2.008842	2.102222	2.374510
0.07	1.339140	1.701540	1.807104	2.015848	2.109401	2.381502
0.08	1.346394	1.708666	1.814139	2.022961	2.116477	2.388371
0.09	1.353632	1.715712	1.821133	2.029747	2.123257	2.394928
0.10	1.360717	1.722577	1.828044	2.036431	2.129964	2.401439
0.15	1.394691	1.755185	1.860267	2.068003	2.160968	2.431694
0.20	1.426029	1.784750	1.889300	2.096069	2.188663	2.458094
0.25	1.454635	1.811324	1.915185	2.120902	2.213067	2.480934
0.30	1.480591	1.834861	1.938216	2.142349	2.233973	2.500336
0.35	1.503898	1.855531	1.958124	2.161018	2.251863	2.516574
0.40	1.524704	1.873564	1.975278	2.176768	2.267009	2.530021
0.45	1.543034	1.889094	1.990116	2.190004	2.279616	2.540934
0.50	1.559094	1.902265	2.002473	2.201200	2.290273	2.549641
0.55	1.573039	1.913487	2.013031	2.210150	2.298575	2.556615
0.60	1.585148	1.922783	2.021584	2.217448	2.305356	2.562057
0.65	1.595441	1.930570	2.028706	2.223336	2.310856	2.566115
0.70	1.604329	1.936988	2.034563	2.228120	2.315141	2.569253
0.75	1.611804	1.942238	2.039192	2.231836	2.318608	2.571621
0.80	1.618075	1.946476	2.042889	2.234565	2.320861	2.573240
0.85	1.623354	1.949947	2.045924	2.236780	2.322569	2.574507
0.90	1.627734	1.952547	2.048190	2.238430	2.324226	2.575317
0.95	1.631392	1.954670	2.049986	2.239752	2.325128	2.575999
1.0	1.634401	1.956423	2.051249	2.240476	2.326339	2.576275
1.1	1.638840	1.958590	2.053171	2.241585	2.327224	2.576799
1.2	1.641727	1.959986	2.054064	2.242350	2.327050	2.577051
1.3	1.643522	1.960541	2.054750	2.242413	2.327984	2.577070
1.4	1.644607	1.960917	2.054949	2.242456	2.327962	2.576978
1.5	1.645308	1.961144	2.059950	2.242557	2.327888	2.576880
1.6	1.645648	1.961166	2.055057	2.242478	2.327804	2.576944
1.7	1.645906	1.961200	2.054881	2.242736	2.327201	2.576940
1.8	1.645948	1.961158	2.055017	2.242478	2.327852	2.576950
1.9	1.645961	1.961119	2.054800	2.242613	2.327199	2.576926
2.0	1.645964	1.961075	2.054768	2.242509	2.327203	2.576712
2.1	1.645909	1.961014	2.054838	2.242351	2.327562	2.576704
2.2	1.645818	1.960899	2.054820	2.242209	2.327690	2.576740
2.3	1.645823	1.960930	2.054787	2.242234	2.327584	2.576644
2.4	1.645719	1.960825	2.054695	2.242143	2.327559	2.576768
2.5	1.645742	1.960853	2.054653	2.242216	2.327176	2.576606
2.6	1.645711	1.960833	2.054470	2.242426	2.326834	2.576642

Table 1B. Continued

2.7	1.645637	1.960791	2.054506	2.242231	2.326950	2.576564
2.8	1.645590	1.960666	2.054598	2.242026	2.327523	2.576594
2.9	1.645570	1.960694	2.054491	2.242087	2.326953	2.576518
3.0	1.645558	1.960707	2.054382	2.242221	2.326884	2.576578
3.1	1.645474	1.960613	2.054549	2.242039	2.327407	2.576532
3.2	1.645488	1.960613	2.054294	2.242132	2.326717	2.576504
$\lambda \downarrow \quad \| \alpha \rightarrow$	0.90	0.95	0.96	0.975	0.98	0.99
3.3	1.645415	1.960518	2.054481	2.241884	2.327434	2.576521
3.4	1.645466	1.960594	2.054265	2.242045	2.326620	2.576445
3.5	1.645430	1.960526	2.054435	2.241953	2.327321	2.576440
3.6	1.645399	1.960585	2.054206	2.242235	2.326654	2.576527
3.7	1.645338	1.960508	2.054390	2.241919	2.327152	2.576455
3.8	1.645341	1.960490	2.054281	2.241911	2.326835	2.576369
3.9	1.645379	1.960450	2.054197	2.242022	2.326782	2.576330
4.0	1.645231	1.960373	2.054328	2.241714	2.327218	2.576309
4.1	1.645315	1.960442	2.054154	2.241898	2.326652	2.576407
4.2	1.645232	1.960439	2.054178	2.241854	2.326687	2.576256
4.3	1.645216	1.960421	2.054332	2.241768	2.327075	2.576212
4.4	1.645209	1.960414	2.054117	2.242002	2.326591	2.576272
4.5	1.645218	1.960400	2.054053	2.241882	2.326596	2.576298
4.6	1.645162	1.960299	2.054246	2.241636	2.327101	2.576333
4.7	1.645162	1.960338	2.054202	2.241751	2.326936	2.576291
4.8	1.645208	1.960351	2.054063	2.241928	2.326533	2.576241
4.9	1.645163	1.960280	2.054060	2.241765	2.326623	2.576122
5.0	1.645114	1.960214	2.054205	2.241575	2.327085	2.576106
5.1	1.645086	1.960246	2.054112	2.241644	2.326876	2.576236
5.2	1.645165	1.960308	2.053991	2.241825	2.326466	2.576260
5.3	1.645135	1.960306	2.053968	2.241907	2.326391	2.576162
5.4	1.645122	1.960253	2.054074	2.241737	2.326709	2.576184
5.5	1.645047	1.960210	2.054136	2.241636	2.326965	2.576106
5.6	1.645074	1.960243	2.054071	2.241744	2.326695	2.576144
5.7	1.645095	1.960286	2.053924	2.241871	2.326429	2.576125
5.8	1.645071	1.960233	2.053909	2.241780	2.326384	2.576143
5.9	1.645000	1.960143	2.053952	2.241593	2.326621	2.576128
6.0	1.645002	1.960103	2.054066	2.241451	2.326907	2.576198
6.1	1.644974	1.960113	2.054026	2.241435	2.326910	2.576147
6.2	1.645003	1.960126	2.053954	2.241558	2.326640	2.576085
6.3	1.645052	1.960183	2.053896	2.241696	2.326429	2.576078
6.4	1.645045	1.960172	2.053872	2.241734	2.326339	2.576060
6.5	1.645009	1.960141	2.053902	2.241628	2.326459	2.576040
6.6	1.644982	1.960095	2.053987	2.241485	2.326643	2.576015
6.7	1.644965	1.960078	2.054038	2.241488	2.326882	2.576034
6.8	1.644951	1.960053	2.053990	2.241437	2.326824	2.576046
6.9	1.644988	1.960112	2.053900	2.241568	2.326596	2.576126
7.0	1.644949	1.960154	2.053835	2.241636	2.326317	2.576030
7.1	1.644964	1.960172	2.053805	2.241699	2.326186	2.575996
7.2	1.644953	1.960166	2.053807	2.241727	2.326238	2.576077
7.3	1.644914	1.960135	2.053827	2.241632	2.326301	2.575969
7.4	1.644974	1.960074	2.053883	2.241616	2.326524	2.576031

Table 1B. Continued

7.5	1.644917	1.960047	2.053996	2.241583	2.326695	2.576053
7.6	1.644905	1.96007	2.053986	2.241468	2.326820	2.575998
7.7	1.644881	1.960032	2.053922	2.241447	2.326658	2.575943
7.8	1.644929	1.960073	2.053898	2.241500	2.326506	2.575923
7.9	1.644862	1.960086	2.053796	2.241544	2.326368	2.575921
8.0	1.644862	1.960086	2.053796	2.241544	2.326368	2.575921
8.1	1.644943	1.960109	2.053767	2.241632	2.326222	2.575927
8.2	1.644900	1.960052	2.053782	2.241569	2.326266	2.575962
8.3	1.644896	1.960029	2.053752	2.241511	2.326375	2.575969
8.4	1.644865	1.959982	2.053821	2.241426	2.326483	2.575973
$\lambda\downarrow$ $\mid\alpha\rightarrow$	0.90	0.95	0.96	0.975	0.98	0.99
8.5	1.644835	1.959963	2.053842	2.241366	2.326642	2.576032
8.6	1.644836	1.959950	2.053918	2.241271	2.326747	2.575987
8.7	1.644781	1.959948	2.053914	2.241361	2.326842	2.576077
8.8	1.644840	1.959959	2.053904	2.241307	2.326759	2.576039
8.9	1.644839	1.959961	2.053845	2.241415	2.326647	2.575979
9.0	1.644861	1.959939	2.053783	2.241389	2.326466	2.575960
9.1	1.644858	1.959983	2.053723	2.241461	2.326388	2.575918
9.2	1.644885	1.960014	2.053756	2.241531	2.326252	2.575928
9.3	1.644882	1.960022	2.053730	2.241538	2.326224	2.575987
9.4	1.644870	1.960017	2.053703	2.241534	2.326170	2.575862
9.5	1.644853	1.960025	2.053738	2.241544	2.326254	2.576012
9.6	1.644877	1.960038	2.053720	2.241512	2.326289	2.575952
9.7	1.644810	1.959976	2.053759	2.241423	2.326382	2.575969
9.8	1.644811	1.959973	2.053800	2.241396	2.326444	2.575875
9.9	1.644769	1.959939	2.053802	2.241360	2.326500	2.575831
10.0	1.644781	1.959948	2.053859	2.241353	2.326638	2.575943
10.5	1.644804	1.959943	2.053758	2.241444	2.326391	2.575953
11.0	1.644804	1.960052	2.053653	2.241601	2.326104	2.575897
11.5	1.644781	1.959947	2.053681	2.241464	2.326186	2.575794
12.0	1.644741	1.959817	2.053787	2.241293	2.326544	2.575793
12.5	1.644716	1.959876	2.053757	2.241254	2.326530	2.575768
13.0	1.644757	1.959929	2.053645	2.241376	2.326241	2.575774
13.5	1.644741	1.959922	2.053582	2.241459	2.326090	2.575783
14.0	1.644732	1.959921	2.053618	2.241399	2.326186	2.575842
14.5	1.644684	1.959809	2.053642	2.241237	2.326385	2.575796
15.0	1.644638	1.959769	2.053717	2.241128	2.326577	2.575814
15.5	1.644644	1.959782	2.053737	2.241101	2.326580	2.575795
16.0	1.644663	1.959782	2.053644	2.241154	2.326394	2.575750
16.5	1.644685	1.959836	2.053550	2.241262	2.326212	2.575741
17.0	1.644674	1.959845	2.053551	2.241346	2.326080	2.575808
17.5	1.644698	1.959858	2.053542	2.241395	2.326020	2.575665
18.0	1.644710	1.959889	2.053565	2.241372	2.326083	2.575678
18.5	1.644645	1.959817	2.053607	2.241333	2.326142	2.575747
19.0	1.644663	1.959800	2.053619	2.241271	2.326221	2.575731
19.5	1.644634	1.959747	2.053661	2.241215	2.326354	2.575740
20.0	1.644617	1.959751	2.053672	2.241223	2.326401	2.575616
20.5	1.644623	1.959762	2.053698	2.241181	2.326468	2.575791

Table 1B. Continued

21.0	1.644602	1.959722	2.053689	2.241216	2.326337	2.575552
21.5	1.644638	1.959779	2.053610	2.241245	2.326266	2.575705
22.0	1.644612	1.959777	2.053609	2.241303	2.326148	2.575632
22.5	1.644621	1.959820	2.053550	2.241317	2.326071	2.575730
23.0	1.644659	1.959839	2.053537	2.241369	2.326041	2.575757
23.5	1.644647	1.959841	2.053499	2.241389	2.325941	2.575648
24.0	1.644614	1.959878	2.053466	2.241425	2.325935	2.575692
24.5	1.644658	1.959882	2.053480	2.241397	2.325907	2.575722
25.0	1.644639	1.959836	2.053463	2.241375	2.325944	2.575668
25.5	1.644635	1.959838	2.053506	2.241361	2.326009	2.575766
26.5	1.644627	1.959805	2.053518	2.241291	2.326085	2.575695
27.0	1.644580	1.959718	2.053498	2.241234	2.326143	2.575725
27.5	1.644571	1.959727	2.053531	2.241205	2.326167	2.575708
28.0	1.644597	1.959732	2.053560	2.241215	2.326225	2.575627
28.5	1.644562	1.959702	2.053602	2.241172	2.326275	2.575609
$\lambda \downarrow \quad \vert \alpha \rightarrow$	0.90	0.95	0.96	0.975	0.98	0.99
29.0	1.644596	1.959695	2.053639	2.241164	2.326395	2.575679
29.5	1.644554	1.959687	2.053625	2.241102	2.326410	2.575647
30.0	1.644567	1.959746	2.053641	2.241064	2.326440	2.575569
31.0	1.644542	1.959689	2.053634	2.241111	2.326441	2.575682
32.0	1.644529	1.959683	2.053624	2.241048	2.326381	2.575607
33.0	1.644538	1.959699	2.053558	2.241121	2.326288	2.575529
34.0	1.644552	1.959725	2.053554	2.241144	2.326236	2.575658
35.0	1.644579	1.959725	2.053534	2.241180	2.326200	2.575612
36.0	1.644555	1.959697	2.053427	2.241170	2.326102	2.575533
37.0	1.644047	1.959700	2.053440	2.241219	2.326077	2.575647
38.0	1.644570	1.959698	2.053447	2.241282	2.326026	2.575629
39.0	1.644555	1.959742	2.053442	2.241288	2.325948	2.575530
40.0	1.644563	1.959744	2.053426	2.241311	2.325948	2.575642
45.0	1.644552	1.959688	2.053419	2.241203	2.326003	2.575680
50.0	1.644524	1.959636	2.053461	2.241076	2.326203	2.575640

Conclusion

For any number of runs, the program will calculate the percentiles of skew-normal distributions for a given value of λ. It is advisable that at least one million runs be used to find percentiles. The above table is created using 100 million runs, so these reported results are more precise. One can expect to have better results if the number of runs is very high. From the table it may be noted that for $\lambda = 1.1,\ \alpha = 0.05$, $Z'_{\alpha}(\lambda) = Z'_{0.05}(1.1) = -0.69448$.

However, if $\lambda = -1.1,\ \alpha = 0.05$, $Z'_{\alpha}(\lambda) = Z'_{0.05}(-1.1)$ is not available in the table. Now the value of $Z'_{\alpha}(\lambda) = Z'_{0.05}(-1.1)$ can be derived as $Z'_{0.05}(-1.1) = -Z'_{0.95}(1.1) = -1.95859$. Let $X' \sim SN(3.0, 1.5, 1.1)$. Then, $X'_{0.05}(1.1)$ can be obtained as $X'_{0.05}(1.1) = 3 + (1.5)\ Z'_{0.05}(1.1)$ $= 3 + (1.5)(0.69448) = 1.95828$, and $X'_{0.05}(-1.1)$ is derived as $X'_{0.05}(-1.1) = 3 - (1.5)\ Z'_{0.95}$ $(1.1) = 3 - (1.5)(1.95859) = 0.062115$.

References

Adcock, C., & Shutes, K. (1999). Portfolio selection based on the multivariate skewed normal distribution, *Financial Modeling, (A Skulimowski edited), Progress and Business Publishers, Krakow*, 35-60.

Aigner, D. J., & Lovell, C. A. K. (1977). Formulation and estimation of stochastic frontier production function model. *Econometrics*, *12*, 21-37.

Arnold, B. C., & Lin, G. D. (2004). Characterization of the skew-normal and generalized Chi distributions. *Sankhya*, *66*(4), 593-606.

Azzalini, A. (1985). A class of distributions which includes the normal ones. *Scandinavian Journal of Statistics, 12*, 171-178.

Azzalini, A. (1986). Further results on a class of distributions which includes the normal ones. *Statistica, 46*, 199-208.

Azzalini, A., & Dalla valle, A. (1996). The multivariate skew-normal distribution. *Biometrika*, *83*(4), 715-726.

Azzalini, A., & Capitanio, A. (1999). Statistical applications of the multivariate skew normal distribution. *Journal of the Royal Statistical Society*, *B, 61*(3), 579-602.

Bartolucci, F., De Luca G., & Loperfido, N. (2000). A generalization for skewness in the basic stochastic volatility model, *Proceedings of the 15th International Workshop on Statistical Modelling,* 140-145.

Box, G. E. P. & Muller, M. E. (1958). A note on the generation of random normal deviates. *Annals of Mathematical Statistics*, *29*, 610-611.

Gupta, A. K., Nguyen, T. T., & Sanqui, J. A. T. (2004a). Characterization of the skew-normal distribution. *Annals of the Institute of Statistical Mathematics*, *56*(2), 351-360.

Gupta, A. K., Gonzalez-Farias, G., & Dominguez-Molina, J. A. (2004b). A multivariate skew normal distribution. *Journal of Multivariate Analysis*, *89*, 181-190.

O'Hagan, A., & Leonard, T. (1982). Bayes estimation subject to uncertainty about parameter constraints. *Biometrika*, *63*, 201-203.

Roberts, C. (1966). A correlation model useful in the study of twins. *Journal of the American Statistical Association, 61*(316), 1184-1190.

Rubinstein, R. Y. (1981). *Simulation and the monte carlo method*. New York, N.Y.: Wiley & Sons.

Appendix

```
Imports System.Math
Public Class Form1
    Inherits System.Windows.Forms.Form

    Dim u1, u2, z1, z2, delta, lamda, zScore, num As Single
    Dim p1, p2, p3, p4, p5, p6, p7, p8, p9, p10, p11, p12 As Single
    Dim i, n As Integer
    Dim file1 As System.IO.StreamWriter

    Private Sub Button1_Click(ByVal sender As System.Object, ByVal e As
System.EventArgs) Handles Button1.Click
        n = Val(TextBox1.Text)
        lamda = Val(TextBox2.Text)
        file1 = System.IO.File.CreateText("C:\file1.txt")

        Dim zScoreArray(n) As Single

        For i = 1 To n
            u1 = Rnd()
            u2 = Rnd()
            z1 = (Sqrt((-2) * Log(u1))) * Cos((360) * u2)
            z2 = (Sqrt((-2) * Log(u1))) * Sin((360) * u2)

            delta = (lamda / (Sqrt(1 + (lamda * lamda))))
            zScore = (delta * Abs(z1) + Sqrt(1 - (delta * delta)) * z2)
            zScoreArray(i) = zScore
        Next
        Array.Sort(zScoreArray)
        p1 = 0.01 * n
        p2 = 0.02 * n
        p3 = 0.025 * n
        p4 = 0.04 * n
        p5 = 0.05 * n
        p6 = 0.1 * n
        p7 = 0.9 * n
        p8 = 0.95 * n
        p9 = 0.96 * n
        p10 = 0.975 * n
        p11 = 0.98 * n
        p12 = 0.99 * n

        TextBox3.Text = zScoreArray(p1)
        TextBox4.Text = zScoreArray(p2)
        TextBox5.Text = zScoreArray(p3)
        TextBox7.Text = zScoreArray(p4)
        TextBox8.Text = zScoreArray(p5)
        TextBox9.Text = zScoreArray(p6)
        TextBox10.Text = zScoreArray(p7)
```

```
TextBox11.Text = zScoreArray(p8)
        TextBox12.Text = zScoreArray(p9)
        TextBox13.Text = zScoreArray(p10)
        TextBox14.Text = zScoreArray(p11)
        TextBox15.Text = zScoreArray(p12)

        file1.WriteLine("Number of Runs = " & n & vbCrLf & vbCrLf _
                        & "Lamda = " & lamda & vbCrLf & vbCrLf _
                        & "0.01 = " & zScoreArray(p1) & vbCrLf _
                        & "0.02 = " & zScoreArray(p2) & vbCrLf _
                        & "0.025 = " & zScoreArray(p3) & vbCrLf _
                        & "0.04 = " & zScoreArray(p4) & vbCrLf _
                        & "0.05 = " & zScoreArray(p5) & vbCrLf _
                        & "0.1 = " & zScoreArray(p6) & vbCrLf _
                        & "0.9 = " & zScoreArray(p7) & vbCrLf _
                        & "0.95 = " & zScoreArray(p8) & vbCrLf _
                        & "0.96 = " & zScoreArray(p9) & vbCrLf _
                        & "0.975 = " & zScoreArray(p10) & vbCrLf _
                        & "0.98 = " & zScoreArray(p11) & vbCrLf _
                        & "0.99 = " & zScoreArray(p12) & vbCrLf & _
                        vbCrLf _ & "Date and Time is " &
                        System.DateTime.Now)

        file1.Close()

    End Sub

    Private Sub Form1_Load(ByVal sender As System.Object, ByVal e As
System.EventArgs) Handles MyBase.Load
        Randomize()     'Calling the random number generator
    End Sub
End Class
```